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Entropy of Schwarzschild black hole and string-black hole correspondence

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Abstract

The *string-black hole* correspondence is considered in the context of the correspondence principle proposed recently by Horowitz and Polchinski [7]. We demonstrate that the entropy of string states and the entropy of a Schwarzschild black hole can be matched including the subleading terms which depend on mass logarithmically. We argue the necessity to include the string interaction (with coupling g) in the consideration and propose the g^2-dependent modification of the string entropy. The matching of it with the entropy of Schwarzschild black hole is analyzed. We also discuss a possible scenario when the entropy of a Schwarzschild black hole appears entirely as effect of the interaction.

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An intriguing idea proposed by Susskind [1] is to identify the states of a black hole with the highly excited states of a fundamental string. A heuristic argument in support of this suggestion is that the number of states for both the hole and the string grows rapidly as a function of mass; and as the string coupling increases, the size of a string state becomes smaller than its Schwarzschild radius, hence any such state must collapse to form a black hole. Recently, remarkable progress has been made towards establishing this correspondence for an extreme black hole: the relevant string states are BPS states characterized by charges identical to those of the extreme black hole [2]. Their entropy is then identical to the Bekenstein-Hawking entropy defined by the horizon area [2] (see review in [3]).

The attempt to extend this correspondence to non-extreme black holes [1], [4], [5], [6] runs into an obvious problem: the number of states of the black hole grows as e^{M^2} while that of the string as e^M. To resolve this discrepancy, Horowitz and Polchinski [7] have recently proposed a correspondence principle: the entropy of string states and the black hole entropy should be matched only for a single (critical) value of the string coupling $g = g_{cr}$, for which the string size becomes of the order of the Schwarzschild radius. Therefore, varying the coupling g from zero to g_{cr} we travel from the *free-string phase* to the phase of the collapsing strings which ends by the formation of the hole.

In this note we make a few comments regarding the *string-black hole* correspondence for Schwarzschild black hole and, in particular, arrive at a conclusion that black hole states can be naturally identified with the states of *perturbatively interacting string* rather than with the *free* states.

We start with considering a free string. The number of its states at the excitation level N is given by [8]

$$d(N) = e^{2\pi\sqrt{\frac{c}{6}N}} N^{-B} \quad , \tag{1}$$

where c is the effective two-dimensional central charge, while mass of the excitation is

$$M_s^2 = \frac{N}{l_s^2} \quad , \tag{2}$$

where l_s is the string size, $l_s \sim \sqrt{\alpha'}$. The exponent B in (1) is universally related to $D_\perp$, the effective number of space-time dimensions for transverse string oscillations (or,

equivalently, the number of uncompactified bosonic degrees of freedom in the worldsheet CFT [9]): $B = \frac{1}{4}(3 + D_\perp)$. We have [8] $B = \frac{27}{4}$ for bosonic string and $B = \frac{11}{4}$ for type-II superstring and heterotic string.

Define a macroscopic state of the string system by fixing the mass M_s (or, equivalently, N in accord with (2)). Then it is realized by $d(N)$ microstates of the string with the entropy

$$
\begin{aligned}
S_s &= \ln d(N) = 2\pi\sqrt{\frac{c}{6}}\sqrt{N} - B\ln N \\
&= 2\pi\sqrt{\frac{c}{6}}(l_s M_s) - 2B\ln(l_s M_s) \quad ,
\end{aligned} \tag{3}
$$

where in the last line we have used (2). We see that the subleading term in (1) gives rise to a subleading term in the string entropy (3) which depends on mass logarithmically. A natural question arises if there exists an analogous term in the entropy in the black hole phase?

A black hole in string theory arises as a solution of the low-energy action which can be isolated as the lowest order term in the expansion

$$
W_{eff} = -\frac{1}{16\pi G}\left(\int_{M^4} R + \int_{M^4} \mathcal{L}(\phi, A_\mu, g_{\mu\nu}, ...)\right) - \sum_{k=0}^{\infty}(gl_s)^{2k} W_k \quad , \tag{4}
$$

where $G = g^2 l_s^2$ is the string induced gravitational constant, the Planck distance l_{pl} is defined as $G = l_{pl}^2$. $\mathcal{L}(\phi, A_\mu, g_{\mu\nu}, ...)$ is the Lagrangian for the (super) multiplet of matter fields appearing in the low-energy approximation. Other terms ($\sim (gl_s)^{2k}$) in the expression (4) are presumably rather complicated non-local functionals. Their influence on a low-energy solution results in some "quantum deformation" of the solution. In other words, they are responsible for the quantum back-reaction effects.

We will be considering here only Schwarzschild black holes. Generally, they are non-extreme black hole geometries characterized by a single dimensional parameter - the Schwarzschild radius r_+. It is related to the black hole mass M_{bh} as $r_+ = 2M_{bh}l_{pl}^2$. Each term in (4) which depends on curvature contributes to the entropy of the hole. The Einstein-like term gives rise to the standard Bekenstein-Hawking expression relating the entropy with the horizon area. In general, the contribution of other terms ($\sim (gl_s)^{2k}$) in (4) to the entropy is not easily recovered. However, for a Schwarzschild hole this

can be accomplished as follows. The Schwarzschild geometry is characterized by a single dimensional parameter r_+, and hence one can apply scaling and dimensionality arguments. They give rise to the entropy of the Schwarzschild black hole which in the leading (with respect to r_+) order has the form:

$$S_{bh} = \pi \frac{r_+^2}{l_{pl}^2} - c_1 \ln(\frac{r_+}{\mu}) \quad , \tag{5}$$

where c_1 is the four-dimensional central charge which comes from the integrated $4D$ conformal anomaly for the zero-mass fields in the theory (4): $\int d^4x\sqrt{g}T^\mu_\mu = c_1$; μ^{-1} is a mass scale. For a massless theory with N_0 scalars, $N_{1/2}$ Majorana fermions, N_1 vectors, $N_{3/2}$ spin-3/2 fermions, N_2 gravitons, and N_A rank-two antisymmetric tensor fields, the coefficient c_1 reads [10], [11]

$$c_1 = \frac{\chi}{90}(-N_0 - \frac{7}{4}N_{1/2} + 13N_1 + \frac{233}{4}N_{3/2} - 212N_2 - 91N_A) \quad , \tag{6}$$

where χ is the Euler number, for the Schwarzschild black hole it is equal to 2.

In the microcanonical ensemble the black hole entropy can be calculated as minus the action functional considered on the Euclidean black hole instanton. Thus, the classical Bekenstein-Hawking entropy in (5) arises from the first (low-energy) term in the effective action (4) while the series $\sum_{k=0}^{\infty}(gl_s)^{2k}W_k$ results in some corrections. The non-local part of the first term in the series, W_0, can be interpreted as due to the one-loop quantization of the low-energy theory. Its contribution to the entropy is isolated by the scaling arguments†. Indeed, considering the quantum part of the effective action (4) on the Schwarzschild black hole instanton, $W_0[g^{sch}_{\mu\nu}(r_+)]$, and performing the rescaling we find

$$W_0[g^{sch}_{\mu\nu}(r_+)] = W_0[\alpha^2 g^{sch}_{\mu\nu}(\frac{r_+}{\alpha})] = W_0[g^{sch}_{\mu\nu}(\frac{r_+}{\alpha})] + \left(\int d^4x\sqrt{g}T^\mu_\mu\right)\ln\alpha$$

that gives $W_0[g^{sch}_{\mu\nu}(r_+)] = c_1 \ln\frac{r_+}{\mu} + const$ and the second term in (5) (an additive constant is omitted in (5)).

It should be noted that the form (5) is quite universal. It appears in different models in two [12], three [13] and four [14] dimensions. Unfortunately, for a charged black hole the scaling arguments are not restrictive enough to recover the form of the entropy since

† The contribution to the entropy of other terms in the series is proportional to $(\frac{gl_s}{r_+})^{2k} \simeq (\frac{1}{gl_sM})^{2k}$, $k = 1, 2, ...$ and is omitted in (5).

there are more than one dimensional parameter characterizing the geometry (see, however, [15]).

A nice thing about (5) is that it already contains the back-reaction effects if the horizon radius r_+ is considered as the radius of the "quantum-corrected" black hole. In terms of the black hole mass Eq.(5) reads[‡]

$$S_{bh} = 4\pi(gl_s M_{bh})^2 - c_1 \ln(l_s M_{bh}) \quad , \tag{7}$$

where we have used $l_{pl} = gl_s$ and omitted term $\sim \ln(\frac{l_s g^2}{\mu})$. This is the entropy in the *black hole phase*. The comparison of it with the entropy in the *free-string phase* (3) shows that both quantities S_s and S_{bh} have similar subleading terms while their leading behavior is considerably different: $S_s \sim M$, $S_{bh} \sim M^2$. In order to resolve this, Horowitz and Polchinski [7] proposed the correspondence principle according to which the two expressions (3) and (7) are matched for a single value $g = g_{cr}$ of the string coupling constant. This value of the coupling corresponds to the transition from the string description to the black hole description. Indeed, as g increases, the string macroscopic state collapses to form a black hole. This is signaled by the size of the string becoming of the order of the Schwarzschild radius, or more precisely, $l_s = \sqrt{\frac{c}{6}} r_+$. This identity defines g_{cr}. Assuming that the mass does not change during the transition, $M_s = M_{bh}$, we find that the black hole forms at the following value of the string coupling

$$g^2_{cr} = \sqrt{\frac{c}{24}} (l_s M)^{-1} \simeq N^{-1/2} \quad . \tag{8}$$

For this value of g the entropies (3) and (7) are equal if the $4D$ and $2D$ central charges are related as $c_1 = 2B$. If so, the *string-black hole* correspondence may have a wider range of validity than originally anticipated.

The relation $c_1 = 2B$ is not automatically satisfied and is a constraint on the low-energy string configuration. As is well known [10], [11] the conformal anomaly in four dimensions vanishes ($c_1 = 0$) for $N = 4,\ 8$ supergravity theories. The non-vanishing contribution is possible from sectors which preserve $N = 2$ space-time supersymmetry [11].

‡In general, the relation between the horizon radius and mass M for a "quantum-corrected" black hole might be the following: $r_+^2 = 4M^2 l_{pl}^4 + \kappa l_{pl}^2 \ln M$ (see last paper in Ref.[12]). This would modify the coefficient in front of the log in (7): $c_1 \to c_1 - \kappa\pi$.

For a heterotic $N = 2$ vacuum the relevant massless spectrum contains [16]: *the gravity multiplet*: the graviton, two gravitinos and a spin-1 Abelian gauge boson (graviphoton) (its contribution to the conformal anomaly is $c_1 = -\frac{11}{6}$); *the vector-tensor multiplet*: the dilaton, the rank-two antisymmetric tensor field, 2 Majorana fermions, and a vector boson (the total $c_1 = -\frac{11}{6}$); *the vector multiplet*: a gauge boson, 2 Majorana fermions, and a complex scalar ($c_1 = \frac{1}{6}$); *the hypermultiplet*: 4 scalars and 2 Majorana fermions ($c_1 = -\frac{1}{6}$). In type-II theories the dilaton resides in the *tensor multiplet* [17] consisting on the antisymmetric tensor, 2 Majorana fermions and 3 scalars ($c_1 = -\frac{13}{6}$)[§]. The members of the vector multiplet are in the adjoint representation of the gauge group G. In general we have n_v vector multiplets and n_h hypermultiplets. The massless spectrum results in the total 4D central charge

$$c_1 = \frac{1}{6}(n_v - n_h - 22)$$

for heterotic string theory and

$$c_1 = \frac{1}{6}(n_v - n_h - 24)$$

for type-II string theory. In order to compare this result to the coefficient in front of the subleading term in (3) we should note that in general the degeneracy of the string states is a product of the degeneracies coming from the right- and left-moving sectors. Therefore, the matching condition for both the type-II and heterotic strings is $c_1 = 11$. It is satisfied if numbers of the vector- and hyper-multiplets are related as

$$\begin{aligned} n_v - n_h &= 88 \quad for \;\; heterotic \;\; strings \\ n_v - n_h &= 90 \quad for \;\; type-II \;\; strings \end{aligned} \tag{9}$$

The compactification of the $D = 10$ heterotic string on a six-dimensional manifold $T^2 \times K_3$ leads (see for example [16]) to the gauge group $G = E_7 \times E_8 \times U(1)^2$. This vacuum has $n_v = dim\ G = 383$ vector multiplets. Then Eq.(9) gives $n_h = 255$ for the number of $N = 2$ hypermultiplets. Such an $N = 2$ configuration (if it is actually realizable) gives us

[§]In four dimensions an antisymmetric tensor is dual to a scalar field (the axion) and thus the *vector-tensor multiplet* is dual to an Abelian vector multiplet. On the other hand, the $N = 2$ *tensor multiplet* is dual to an $N = 2$ *hypermultiplet*. The conformal anomaly, however, depends on the field representation and differs for the dual multiplets [10].

an example when the logarithmic term in the black hole entropy (5) has statistical origin as due to the subleading behavior of the number of string states (3). However, we should note that the coefficient c_1 in the black hole calculation seems to depend on the compactification while the coefficient in the string calculation does not. A possible resolution of this is that the matching condition gives us a constraint on the compactification. On other hand, if the coefficients are not matched we still have agreement between two descriptions which now happens for the string coupling taking the critical value modified by the logarithmic term:

$$g^2_{cr} = \sqrt{\frac{c}{24}} \frac{1}{(l_s M)^2} (l_s M + \sigma \ln(l_s M)) \quad , \tag{10}$$

where $\sigma = \sqrt{\frac{24}{c}} \frac{1}{2\pi} (c_1 - 2B)$.

It should be noted that in the above consideration we were matching two quantities S_s and S_{bh} which are defined for essentially different values of g. Indeed, S_s is a free-string quantity ($g = 0$), while g is supposed to be non-zero in defining S_{bh}. Extrapolating S_s to non-zero values of g we get the critical value $g_{cr} \sim N^{-1/4}$. However, there must be an intermediate *gravitational phase* between the *free-string phase* and the *black hole phase*, when the string states are already attracting but the black hole is not yet formed. This phase is characterized by the appearance of the new scale [18] playing the role of an "order parameter". This is, of course, the Planck scale $l_{pl} = g l_s$. In this phase the string interaction can be considered perturbatively that results in a g-dependent string entropy S^g_s. It seems reasonable that namely states of the interacting string may form the hole and in the critical point (when the string and black hole descriptions coincide) we must match the black hole entropy S_{bh} and the interacting string entropy S^g_s (instead of the free quantity S_s). In principle, the string entropy S^g_s can possess some non-perturbative corrections, behaving as $\sim \frac{1}{g^2}$, which are hard to reveal. Therefore, in what follows we consider the string entropy S^g_s as a perturbation series with respect to g and discuss its matching to S_{bh}.

We know that the quantities (1)-(3) are valid for free string, $g = 0$. When the string interaction is turned on there must appear g-dependent corrections to these formulas. It is reasonable to expect that these corrections are controlled by g^2. Therefore, by dimensional arguments [19] we obtain that such a correction to the entropy is determined

by the quantity $(gl_sM_s)^2$. With this correction included, the expected expression for the entropy of the string states for $g \neq 0$ is

$$S_s^g = 2\pi\sqrt{\frac{c}{6}}(l_sM_s) + 4\pi a(gl_sM_s)^2 - 2B\ln\Big((l_sM_s) + 4\pi a(gl_sM_s)^2\Big) \quad , \tag{11}$$

where the value of coefficient a can in principle be determined by a precise calculation. The perturbatively interacting string presumably has the same level structure as the free string. Therefore, the formula (1) is still valid for $g \neq 0$. However, the mass formula (2) for the excited states gets modified as $\sqrt{N} = l_sM_s + a'(gl_sM_s)^2$ ($a = \sqrt{\frac{c}{24}}a'$). These result in the Eq.(11). For small g we find that $l_sM_s^g \simeq \sqrt{N} - a'g^2N$. This agrees with the estimation made in [6]. The coefficient a in (11) must be positive to ensure positiveness of the entropy for large M_s ¶.

The matching of (11) and (7) can be done only if $0 \leq a < 1$. This means that for large M the black hole entropy S_{bh} must grow faster than S_s^g that is sensible in the spirit of the second law. The matching condition gives the following critical value for the string coupling

$$g_{cr}^2 = \frac{1}{(1-a)}\frac{1}{(l_sM_s)}\sqrt{\frac{c}{24}} \quad . \tag{12}$$

Equation (12) indicates that at the matching point the correction term in (11) becomes important and can not be neglected. This is especially true if the coefficient a in (11) is slightly different than 1 [20]. The critical Schwarzschild radius is $r_+^{cr} = 2g_{cr}^2l_s^2M = \frac{1}{1-a}\sqrt{\frac{c}{6}}l_s$, and for $a \to 1$ we have that g^{cr}, $r_+^{cr} \to \infty$. This means that the case $a = 1$ is special. Indeed, the entropies (11) and (7) can not be matched for any finite g if $a = 1$. However, in this case the ($\sim g^2$) term in Eq.(11) exactly reproduces the corresponding term in the black hole entropy (7) which can be identified by equation

$$S_{bh} = S_s^g - S_s^{g=0} \tag{13}$$

valid in the point of the transition. Note that in arbitrary dimension $d \geq 4$ the black hole entropy behaves as $S_{bh}^{(d)} \simeq g^{\frac{2}{d-3}}(l_sM)^{\frac{d-2}{d-3}}$. Therefore, only in four dimensions it grows as an integer power of the string coupling g that makes the interpretation (13) possible.

¶ Other reason for this is that the derivative $\frac{dS_s}{dg} \geq 0$ that means validity of the second law in the process of collapse of the string states.

Thus, in this scenario ($a = 1$) the black hole entropy appears as purely *string-loop* effect due to the self-interaction of the fundamental string. Therefore, it is natural to identify the "internal states of black hole" not with states of the *free string* but with (a part of) states of the *interacting string*, and the black hole entropy arises entirely due to the interaction. The total number of the states of the interacting string is bigger (13) than it is required by the black hole entropy. Therefore, not all the string states may collapse and form a black hole (otherwise, we would arrive at a situation when the second law is violated). Some of them (with the entropy $S_s^{g=0}$) must stay outside the horizon and be accessible for an external "observer". They form gas of massless excitations (waves) freely propagating in a black hole.

It is worth noting that an analogous situation happens when we are trying to calculate the quantum field theoretical entropy of a black hole. The quantum entropy arises as a sum of a contribution due to the black hole and the entropy of the hot gas of quantum fields propagating outside the horizon. The contribution of the gas can be isolated by its dependence on the size L of the system (see, for example, [15]). To extract the entropy of the hole itself one should compare the whole quantum entropy with the entropy of flat space filled by the hot gas. Possibly, the similar line of reasoning can be useful in the case under consideration. Then the contribution of flat space is due to the non-interacting string states with the entropy $S_s^{g=0}$.

Our analysis, in principle, does not prohibit the appearance of the higher order terms in the expansion (11). Moreover, these terms are very likely to appear in the regime when $a \sim 1$ and g_{cr} becomes large. Formally adding [20] term $b(gl_sM)^4$ to the string entropy we find that the matching condition leads to some constraint on mass of the string forming black hole. This happens because for sufficiently large mass the interacting string entropy (with higher order corrections added) becomes greater than the black hole entropy and the two quantities can not be matched.

In an alternative scenario the string interaction appears in such a way that the higher order corrections to (11) do not arise and the coefficient a never becomes equal to 1 and changes in the limits $0 \leq a < 1$ only. All next g-dependent corrections to the string entropy result in a modification of the coefficient a in (11) so that it becomes a function

of the string coupling g and runs to 1 when g goes to infinity (Eq.(12) is an indication of this behavior) Then, all the string states can form a black hole, and the matching of the string and black hole entropies always can be done in accord to the correspondence principle. At the moment, we do not have arguments in favor of any of these scenarios, $a > 1$, $a = 1$ or $0 \leq a < 1$, and hope that further investigation will shed light on this problem.

The above analysis of the *string-black hole* correspondence does not say us where and in which form the string states counted by the black hole entropy present in the phase of the macroscopic heavy black hole. This is, however, the most important question regarding the statistical explanation of the black hole entropy. It is intuitively clear that thin ($\sim gl_s$) layer around the horizon is likely to be a place for storing such states [1]. That is why the Bekenstein-Hawking entropy relates their number to geometry of the horizon. From the analysis presented above follows that the *interaction* of the fundamental string is important for the correct description of the *string-black hole* transition. Therefore, we speculate that in the *black hole phase* the states in the layer near horizon are some *bound states* due to the strong interaction of the string modes with the horizon. These states arise from the modes of the perturbatively interacting string in the *string-black hole* transition. This picture resembles that of given in [1].

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[15] For a charged black hole characterized by lower (r_-) and outer (r_+) horizon radii we in general have instead of (5): $S_{bh} = \pi \frac{r_+^2}{l_{pl}^2} + \alpha(\frac{r_-}{r_+}) - \beta(\frac{r_-}{r_+}) \ln r_+$, where $\beta(0) = c_1$. In the extreme limit $r_+ \to r_-$ the entropy again takes the form (5) with c_1 replaced by $\beta(1)$. A concrete calculation see in V. P. Frolov, W. Israel and S. N. Solodukhin, Phys. Rev. **D54**, 2732 (1996).

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[19] Since g is dimensionless there may be corrections to the string entropy given by the series $\alpha_0 + \alpha_1 g + \alpha_2 g^2 + ...$ with coefficients α_k not dependent on parameters (mass M) of the system. Such corrections result in an additive constant to the entropy and are not of our interest. The corrections which are essential for us arise when g comes in the combination with l_s, gl_s. This perturbation series goes with respect to the dimensionless parameter (gl_sM) as shown in (11).

[20] Strictly speaking, in this point the approximation (11) is not sufficient and there may appear higher order corrections. Formally adding the term $b(gl_sM)^4$ to the expression (11) we find that the critical value of the string coupling becomes $g_{cr}^2 = \frac{1}{2b(l_sM)^2}((1-a) - \sqrt{(1-a)^2 - 8\pi b\sqrt{c/6}(l_sM)})$. This gives constraint on the mass $(l_sM) \leq \sqrt{\frac{6}{c}}\frac{(1-a)^2}{8\pi b}$. We see that the case $a = 1$ is special even if the higher order corrections are included.